James V. Steele and John W. Negele
Center for Theoretical Physics, Laboratory for Nuclear Science, and Department of Physics
Massachusetts Institute of Technology, Cambridge, MA 02139, USA
(July 7, 2000)

Merons, conjectured as a semiclassical mechanism for color confinement in QCD, have been described analytically by either infinite action configurations or an *Ansatz* with discontinuous action. We construct a smooth, finite action, stationary lattice solution corresponding to a meron pair. We also derive an analytical solution for the zero mode of the meron pair *Ansatz*, show that it has the qualitative behavior of the exact zero mode of the lattice solution, and propose the use of zero modes to identify meron gauge field configurations in stochastic evaluations of the lattice QCD path integral.

Chiral symmetry breaking and confinement of color charge, the two essential features of low-energy QCD, cannot be understood perturbatively in the coupling constant. Two complementary nonperturbative approaches are analytical approximation using semiclassical analysis of the QCD path integral (analogous to the WKB approximation in quantum mechanics) and numerical solution using lattice gauge theory. Combining physical understanding from the semiclassical approximation with systematic rigor of the lattice has provided valuable insight into hadron structure [1] and our goal is to use this combined approach to study confinement.

Most semiclassical treatments of QCD have focused on instantons, leading to a qualitative, and in some cases even quantitative, understanding of chiral symmetry breaking and the structure of the QCD vacuum [2]. However, simple configurations of instantons, such as a dilute gas [3] or a random superposition [4], do not confine color charges. Of alternative semiclassical mechanisms proposed for confinement [3,5,6], we focus on merons [3], which are solutions to the classical Yang-Mills field equations with one-half unit of localized topological charge. They are the (3+1)-dimensional nonabelian extension of the 't Hooft–Polyakov monopole configuration, which has been shown to lead to confinement in 2+1 dimensions [7].

A purely analytic study of merons has proven intractable for several reasons. Unlike instantons, no exact solution exists for more than two merons [8], gauge fields for isolated merons fall off too slowly ($A \sim 1/x$) to superpose them, and the field strength is singular. A patched *Ansatz* configuration that removes the singularities [3] does not satisfy the classical Yang-Mills equations, preventing even calculation of gaussian fluctuations around a meron pair [9].

This letter lays the groundwork for using lattice techniques to investigate the presence and role of merons in QCD. We construct a smooth, finite action configuration corresponding to a meron pair, which with its associated quantum fluctuations, should be present in Monte Carlo lattice QCD calculations. Similar work with instantons has shown fermionic zero modes are a powerful tool for identifying topological objects on the lattice. Therefore, we also analytically derive the zero mode of the continuum meron pair *Ansatz* and compare with exact numerical results. For simplicity, we restrict our discussion to SU(2) color without loss of generality.

Two known solutions to the classical Yang-Mills equations in four Euclidean dimensions are instantons and merons. Both have topological charge, can be interpreted as tunneling solutions, and can be written in the general form (for the covariant derivative $D_\mu = \partial_\mu - iA^a_\mu \sigma^a/2$)

$$A^a_\mu(x) = \frac{2\,\eta_{a\mu\nu}\,x^\nu}{x^2} f(x^2) \;, \tag{1}$$

with $f(x^2) = x^2/(x^2+\rho^2)$ for an instanton and $f(x^2) = \frac{1}{2}$ for a meron.

Conformal symmetry of the classical Yang-Mills action, in particular under inversion $x_\mu \to x_\mu/x^2$, shows that in addition to a meron at the origin, there is a second meron at infinity. These two merons can be mapped to arbitrary positions, which we define to be the origin and d_μ. After a gauge transformation, the gauge field for the two merons takes the simple form [10]

$$A^a_\mu(x) = \eta_{a\mu\nu}\left[\frac{x^\nu}{x^2} + \frac{(x-d)^\nu}{(x-d)^2}\right] \;. \tag{2}$$

Similar to instantons [2], a meron pair can be expressed in singular gauge by performing a large gauge transformation about the mid-point of the pair, resulting in a gauge field that falls off faster at large distances ($A \sim x^{-3}$). A careful treatment of the singularities shows that the topological charge density is [10]

$$Q(x) \equiv \frac{\mathrm{Tr}}{16\pi^2}\left(F_{\mu\nu}\widetilde{F}^{\mu\nu}\right) = \frac{1}{2}\,\delta^4(x) + \frac{1}{2}\,\delta^4(x-d) \;, \tag{3}$$

yielding total topological charge $Q = 1$, just like the instanton.

The gauge field Eq. (2) has infinite action density at the singularities $x_\mu = \{0, d_\mu\}$. Hence, a finite action *Ansatz* has been suggested [3]

$$A^a_\mu(x) = \eta_{a\mu\nu} x^\nu \begin{cases} \dfrac{2}{x^2+r^2} \;, & \sqrt{x^2} < r \;, \\[2mm] \dfrac{1}{x^2} \;, & r < \sqrt{x^2} < R \;, \\[2mm] \dfrac{2}{x^2+R^2} \;, & R < \sqrt{x^2} \;. \end{cases} \tag{4}$$

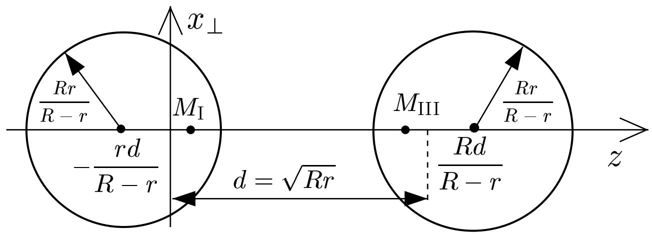

FIG. 1. Meron pair separated by $d = \sqrt{Rr}$ regulated with instanton caps, each containing $\frac{1}{2}$ topological charge.

Here, the singular meron fields for $\sqrt{x^2} < r$ and $\sqrt{x^2} > R$ are replaced by instanton caps, each containing topological charge $\frac{1}{2}$ to agree with Eq. (3). The radii r and R are arbitrary and we will refer to the three separate regions defined in Eq. (4) as regions I, II, and III, respectively. The action for this configuration is

$$S = \frac{8\pi^2}{g^2} + \frac{3\pi^2}{g^2} \ln \frac{R}{r} \ , \tag{5}$$

which shows the divergence in the $r \to 0$ or $R \to \infty$ limit. There is no angular dependence in this patching, and so the conformal symmetry of the meron pair is retained. Although this patching of instanton caps is continuous, the derivatives are not, and so the equations of motion are violated at the boundaries of the regions in Eq. (4).

Applying the same transformations used to attain Eq. (2) to the instanton cap solution, regions I and III become four-dimensional spheres each containing half an instanton [3]. The geometry of the instanton caps is shown in Fig. 1 for the symmetric choice of displacement $d = \sqrt{Rr}$ along the z-direction. Note that the action for the complicated geometry of Fig. 1 is still given by Eq. (5), which for $d = \sqrt{Rr}$ can be rewritten as

$$S = S_0 \left(1 + \frac{3}{4} \ln \frac{d}{r} \right) \ , \tag{6}$$

with $S_0 = 8\pi^2/g^2$. We will concentrate on this case below, and generalization to a different d is straightforward.

The original positions of the two merons $x_\mu = \{0, d_\mu\}$ are not the centers of the spheres, nor are they the positions of maximum action density, which occurs within the spheres with $S_{\rm max} = (48/g^2)(R+r)^4/d^8$ at

$$(M_{\rm I})_\mu = \frac{r^2}{r^2+d^2}\, d_\mu \ , \qquad (M_{\rm III})_\mu = \frac{R^2}{R^2+d^2}\, d_\mu \ . \tag{7}$$

However, in the limit $r \to 0$ and $R \to \infty$ holding d fixed, the spheres will shrink around the original points reducing to the bare meron pair in Eq. (2). In the opposite limit $R \to r$, the radii of the spheres increase to infinity, leaving an instanton of size $\rho = d$.

An instanton can therefore be interpreted as consisting of a meron pair. This complements the fact that instantons and antiinstantons have dipole interactions between each other. If there exists a regime in QCD where meron entropy contributes more to the free energy than the logarithmic potential between pairs, like the Kosterlitz–Thouless phase transition, instantons could break apart into meron pairs [3]. The intrinsic size of the instanton caps would then be determined by the instanton scale ρ.

The Atiyah-Singer index theorem states that a fermion in the presence of a gauge field with topological charge Q, described by the equation

$$\not{D}\, \psi = \lambda \psi \ , \quad \text{with} \quad \psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \ , \tag{8}$$

has n_R right-handed and n_L left-handed zero modes (defined by $\lambda = 0$) such that $Q = n_L - n_R$. This can sometimes be strengthened by a vanishing theorem [11]. Applying $\not{D}$ twice to ψ decouples the right- and left-hand components, and focusing on the equation for ψ_R, gives

$$\left(D^2 + \frac{1}{2} \bar{\eta}_{a\mu\nu} \sigma^a F^{\mu\nu} \right) \psi_R = 0 \ , \tag{9}$$

where $\bar{\eta}_{a\mu\nu}$ denotes the 't Hooft symbol, σ^a acts on the spin indices of ψ_R, and $F^{\mu\nu}$ acts on the color indices. For a self-dual gauge field (like an instanton), the second term in Eq. (9) is zero; and since D^2 is a negative definite operator, there are no normalizable right-handed zero modes, implying $Q = n_L$. Although the meron pair is not self-dual, the second term can be shown to be negative definite as well, leading to the same conclusion.

In general, a gauge field in Lorentz gauge with $Q = 1$ can be written in the form

$$A^a_\mu(x) = -\eta_{a\mu\nu} \partial_\nu \ln \Pi(x) \ . \tag{10}$$

This has a fermion zero mode given by

$$\psi = \begin{pmatrix} 0 \\ \phi \end{pmatrix} \ , \qquad \phi^a_\alpha = \mathcal{N}\, \Pi^{3/2}\, \varepsilon^a_\alpha \ , \tag{11}$$

with normalization $\mathcal{N}$ and $\varepsilon = i\sigma_2$ coupling the spin index α to the color index a (both of which can be either 1 or 2) in a singlet configuration. The gauge field for the meron pair with instanton caps corresponds to

$$\Pi(x) = \begin{cases} \dfrac{2\xi_i d^2}{x^2 + \xi_i^2 (x-d)^2} \ , & \text{for regions } i = \text{I, III}, \\[2ex] \dfrac{d^2}{\sqrt{x^2 (x-d)^2}} \ , & \text{region II}, \end{cases} \tag{12}$$

with $\xi_{\rm I} = r/d$ and $\xi_{\rm III} = R/d$. The normalized solution to the zero mode is then Eq. (11) with

$$\mathcal{N}^{-1} = 2\pi d^2 \left[2 - \sqrt{\frac{r}{R}} \right]^{1/2} \ . \tag{13}$$

Note that the unregulated meron pair ($r \to 0$, $R \to \infty$) has a normalizable zero mode itself [12]

$$\phi^a_\alpha(x) = \frac{d\, \varepsilon^a_\alpha}{2\sqrt{2}\pi \left(x^2 (x-d)^2 \right)^{3/4}} \ . \tag{14}$$

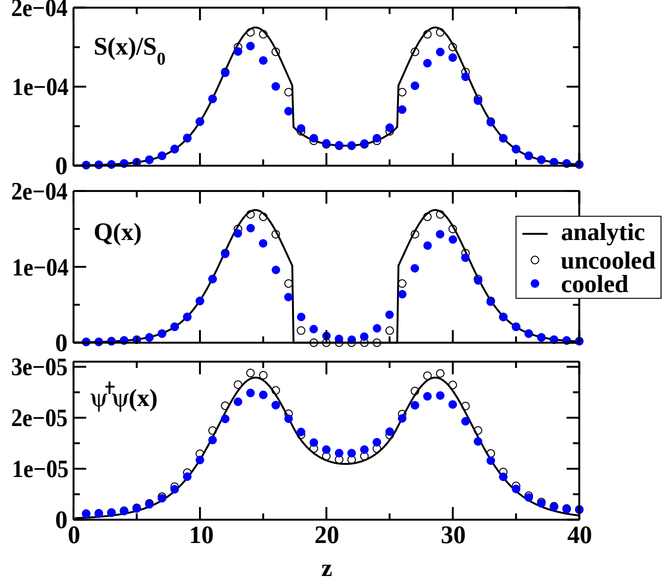

FIG. 2. The action density $S(x)$ for the regulated meron pair, normalized by the instanton action $S_0 = 8\pi^2/g^2$, sliced through the center of the configuration in the direction of separation. Shown are the analytic (solid), initial lattice (open circles) and cooled lattice (filled circles) configurations. The same is shown for the topological charge density $Q(x)$ and fermion zero mode density $\psi^\dagger\psi(x)$.

The gauge-invariant zero mode density $\psi^\dagger\psi(x)$ has a bridge between two merons that falls off like x^{-3} in contrast to the x^{-6} fall-off in all other directions. This behavior can be used to identify merons when analyzing their zero modes on the lattice, similar to what was done for instantons in Ref. [13].

As mentioned above, the patching of instanton caps to obtain an explicit analytic solution has unavoidable and unphysical discontinuities in the action density. Therefore, in order to study this solution further, we put the gauge field on a space-time lattice of spacing a in a box of size $L_0 \times L_1 \times L_2 \times L_3$. The gauge-field degrees of freedom are replaced by the usual parallel transport link variable, $U_\mu(x) = \mathrm{P}\exp\left[-i\int_x^{x+a\,\hat{e}_\mu} A_\nu(z)dz^\nu\right]$. The exponentiated integral can be performed analytically within the instanton caps, producing arctangents. Outside of the caps, the integral is evaluated numerically by dividing each link into as many sub-links as necessary to reduce the $\mathcal{O}(a^3)$ path-ordering errors below machine precision.

We then calculate the action density $S(x)$ for a meron pair with instanton caps using the improved action of Ref. [14] and the topological charge density $Q(x)$ using products of clovers. These (open circles) are compared with the patched *Ansatz* results (solid line) in Fig. 2 for $r = 9$ and $d = 20.8$ (in units of the lattice spacing) on a $32^3 \times 40$ lattice. We use the Arnoldi method to solve for the zero mode of this gauge configuration and hence the density $\psi^\dagger\psi(x)$, which is also compared with the analytic result in the same figure, showing excellent agreement.

Two important features of the patched *Ansatz* also evident with the lattice representation are the discontinuities in the action density at the boundary of the instanton caps and vanishing of the topological charge density outside of the caps as given by Eq. (3). The discontinuity in the action density is unphysical and can be smoothed out by using a relaxation algorithm to iteratively minimize the lattice action [15]. On a sweep through the lattice, referred to as a cooling step, each link is chosen to locally minimize the action density. Since a single instanton is already a minimum of the action, this algorithm would leave the instanton unchanged (for a suitably improved lattice action). The result for the regulated meron pair after ten cooling steps is represented in Fig. 2 by the filled circles, showing the discontinuities in the action density have already been smoothed out. In Fig. 3, we also plot side-by-side the cooled $Q(x)$ and $\psi^\dagger\psi(x)$ in the (z,t)-plane for three different meron pair separations. As the separation vanishes, the zero mode goes over into the well-known instanton zero mode result.

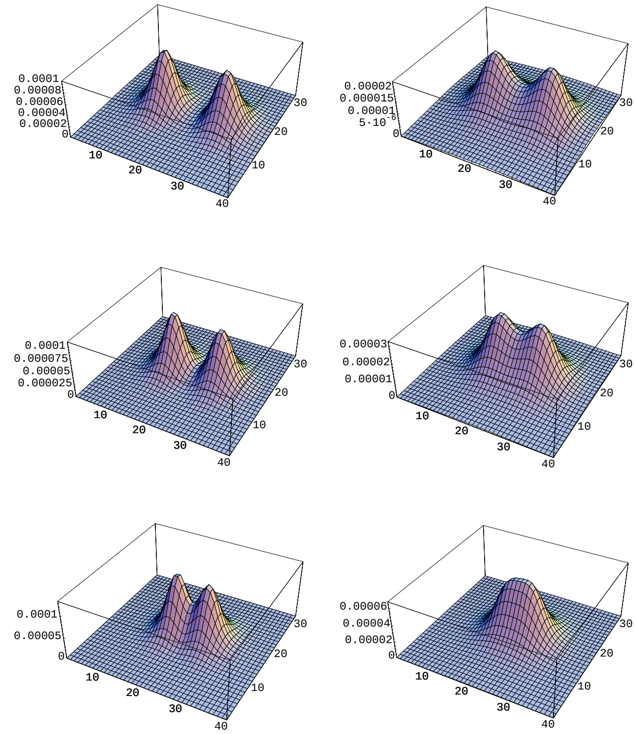

FIG. 3. The topological charge density $Q(x)$ on the left and fermion zero mode density $\psi^\dagger\psi(x)$ on the right for a cooled meron pair with $r = 9$, separated by distance $d = 20.8$, 17.7, and 14.1 respectively, in the (z,t)-plane.

Like instanton-antiinstanton pairs, however, a meron pair is not a strict minimum of the action, since it has a weak attractive interaction, Eq. (5), and under repeated relaxation will coalesce to form an instanton. Nevertheless, just as it is important to sum all the quasi-stationary instanton-antiinstanton configurations to obtain essential nonperturbative physics [16], meron pairs may be expected to play an analogous role. A precise framework for including quasi-stationary meron pairs is to introduce a constraint $\mathcal{Q}[A]$ on a suitable collective variable q (which we chose to be the quadrupole moment $3z^2 - x^2 - y^2 - t^2$ of the topological charge) as follows

$$Z = \int DA \, \exp\{-S[A]\} = \int dq \, Z_q \, , \tag{15}$$

$$Z_q = \int DA \, \exp\left\{-S[A] - \lambda\left(\mathcal{Q}[A] - q\right)^2\right\} \, . \tag{16}$$

The meron pair is then a true minimum of the effective action with constraint, allowing for a semiclassical treatment of Z_q. Afterwards, q is integrated to obtain the full partition function Z.

The criterion for a good collective variable q of the system is that the gradient in the direction of q is small compared to the curvature associated with all the quantum fluctuations. In this case, we have an adiabatic limit in which relaxation of the unconstrained meron pair slowly evolves through a sequence of quasi-stationary solutions, each of which is close to a corresponding stationary constrained solution. Detailed comparison of the quasi-stationary and constrained solutions shows this adiabatic limit is well satisfied. Therefore, we will present here the action of a meron pair as it freely coalesces into an instanton. To compare with the patched *Ansatz*, we need the meron separation d and radii r and R of the lattice configuration. We find these by first measuring the separation of the two maxima in the action density $\Delta \equiv |M_{\rm I} - M_{\rm III}|$ (using cubic splines) and their values (which are the same in the symmetric case, denoted by $S_{\max} = 48/g^2 w^4$), and then using Eq. (7) to give

$$d^2 = \Delta^2 + 4w^2 \, , \qquad \{r, R\} = \frac{2wd}{d \pm \Delta} \, . \tag{17}$$

In Fig. 4, we plot the total action of a meron pair as a function of d/r (solid line), which for the analytic case is given by Eq. (6). Also shown are cooling trajectories for four lattice configurations with different initial meron pair separations. Each case starts with a patched *Ansatz* of a given separation and the lattice action agrees with that of the analytic *Ansatz*.

The first few cooling steps primarily decrease the action without changing the collective variable (which is now effectively d/r), as observed above in Fig. 2. Further cooling gradually decreases the collective variable, tracing out a new logarithmic curve for the total action (dashed line in Fig. 4), which is about 0.25 lower than the analytic case. This curve represents the total action of the smooth adiabatic or constrained lattice meron pair solutions. The essential property of merons that could allow them to dominate the path integral and confine color charge is the logarithmic interaction Eqs. (5,6) which is weak enough to be dominated by the meron entropy [3]. Hence, the key physical result from Fig. 4 is the fact that smooth adiabatic or constrained lattice meron pair solutions clearly exhibit this logarithmic behavior.

In summary, we have found stationary meron pair solutions, shown that they have a logarithmic interaction, and shown that they have characteristic fermion zero modes localized about the individual merons. Hence the presence and role of merons in numerical evaluations of the QCD path integral should be investigated, and the study of fermionic zero modes and cooling of gauge configurations provide possible tools to do so.

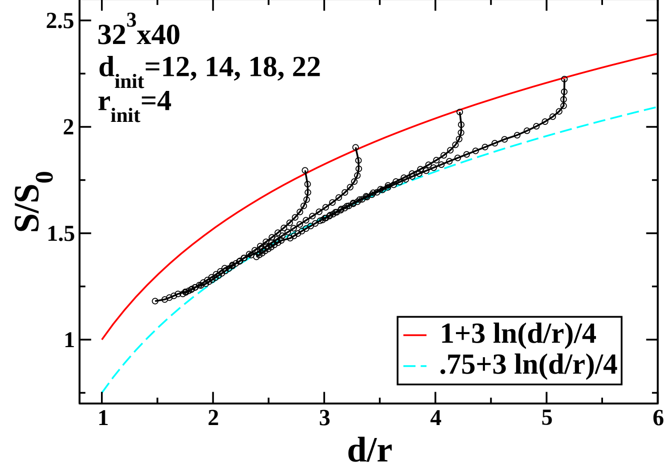

FIG. 4. Action of a regulated meron pair as a function of d/r for initial lattice configurations of $d_{\rm init} = 12, 14, 18, 22$ and $r_{\rm init} = 4$ on a $32^3 \times 40$ lattice. The circles show cooling trajectories with each cooling step marked.

We would like to thank J. Kuti and F. Lenz for useful discussions. This work was supported in part by the U.S. Department of Energy under cooperative research agreement #DF-FC02-94ER40818.

[1] M. C. Chu, J. M. Grandy, S. Huang and J. W. Negele, Phys. Rev. **D49**, 6039 (1994).

[2] T. Schäfer and E. V. Shuryak, Rev. Mod. Phys. **70**, 323 (1998).

[3] C. G. Callan, R. Dashen and D. J. Gross, Phys. Lett. **B66**, 375 (1977); Phys. Rev. **D17**, 2717 (1978); Phys. Rev. **D19**, 1826 (1979).

[4] D. Chen, R. C. Brower, J. W. Negele and E. Shuryak, Nucl. Phys. Proc. Suppl. **73**, 512 (1999).

[5] A. Gonzalez-Arroyo and P. Martinez, Nucl. Phys. **B459**, 337 (1996).

[6] M. Faber, J. Greensite, S. Olejnik and D. Yamada, hep-lat/9912002.

[7] A. M. Polyakov, Nucl. Phys. **B120**, 429 (1977).

[8] A. Actor, Rev. Mod. Phys. **51**, 461 (1979).

[9] D. G. Laughton, Can. J. Phys. **58**, 845, 859 (1980).

[10] V. de Alfaro, S. Fubini and G. Furlan, Phys. Lett. **B65**, 163 (1976).

[11] R. Jackiw and C. Rebbi, Phys. Rev. **D16**, 1052 (1977).

[12] K. Konishi and K. Takenaga, hep-th/9911097.

[13] T. L. Ivanenko and J. W. Negele, Nucl. Phys. Proc. Suppl. **63**, 504 (1998). T. Ivanenko, MIT Ph.D. disseration, 1997.

[14] M. Garcia Perez, A. Gonzalez-Arroyo, J. Snippe and P. van Baal, Nucl. Phys. **B413**, 535 (1994).

[15] B. Berg, Phys. Lett. **B104**, 475 (1981).

[16] S. Coleman, *Aspects of Symmetry*, (Cambridge University Press, 1988).